

# ON FILTERED MULTIPLICATIVE BASES OF GROUP ALGEBRAS II

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*Dedicated to Professor P.M. Gudiavok on his 65th birthday*

**ABSTRACT.** We give an explicit list of all  $p$ -groups  $G$  with a cyclic subgroup of index  $p^2$ , such that the group algebra  $KG$  over the field  $K$  of characteristic  $p$  has a filtered multiplicative  $K$ -basis. We also proved that such a  $K$ -basis does not exist for the group algebra  $KG$ , in the case when  $G$  is either a powerful  $p$ -group or a two generated  $p$ -group ( $p \neq 2$ ) with a central cyclic commutator subgroup. This paper is a continuation of the related [2].

**1. Introduction.** Let  $A$  be a finite-dimensional algebra over a field  $K$  and let  $B$  be a  $K$ -basis of  $A$ . Suppose that  $B$  has the following properties:

1. if  $b_1, b_2 \in B$  then either  $b_1 b_2 = 0$  or  $b_1 b_2 \in B$ ;
2.  $B \cap \text{rad}(A)$  is a  $K$ -basis for  $\text{rad}(A)$ , where  $\text{rad}(A)$  denotes the Jacobson radical of  $A$ .

Then  $B$  is called a *filtered multiplicative  $K$ -basis* of  $A$ .

The filtered multiplicative  $K$ -basis arises in the theory of representations of algebras and was first introduced by H. Kupisch [5]. In [1] R. Bautista, P. Gabriel, A. Roiter and L. Salmeron proved that if there are only finitely many isomorphism classes of indecomposable  $A$ -modules over an algebraically closed field  $K$ , then  $A$  has a filtered multiplicative  $K$ -basis. Note that by Higman's theorem the group algebra  $KG$  over a field of characteristic  $p$  has only finitely many isomorphism classes of indecomposable  $KG$ -modules if and only if all the Sylow  $p$ -subgroups of  $G$  are cyclic.

Here we study the following question from [1]: *When does a filtered multiplicative  $K$ -basis exist in the group algebra  $KG$ ?*

Let  $G$  be a finite abelian  $p$ -group. Then  $G = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$  is the direct product of cyclic groups  $\langle a_i \rangle$  of order  $q_i$ , the set

$$B = \{(a_1 - 1)^{n_1} (a_2 - 1)^{n_2} \cdots (a_s - 1)^{n_s} \mid 0 \leq n_i < q_i\}$$

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is a filtered multiplicative  $K$ -basis of the group algebra  $KG$  over the field  $K$  of characteristic  $p$ .

Moreover, if  $KG_1$  and  $KG_2$  have filtered multiplicative  $K$ -bases, which we denote by  $B_1$  and  $B_2$  respectively, then  $B_1 \times B_2$  is a filtered multiplicative  $K$ -basis of the group algebra  $K[G_1 \times G_2]$ .

P. Landrock and G.O. Michler [6] proved that the group algebra of the smallest Janko group over a field of characteristic 2 does not have a filtered multiplicative  $K$ -basis.

L. Paris [9] gave examples of group algebras  $KG$ , which have no filtered multiplicative  $K$ -bases. He also showed that if  $K$  is a field of characteristic 2 and either a)  $G$  is a quaternion group of order 8 and also  $K$  contains a primitive cube root of the unity or b)  $G$  is a dihedral 2-group, then  $KG$  has a filtered multiplicative  $K$ -basis. We showed in [2] that for the class of all metacyclic  $p$ -groups the groups mentioned in the items a) and b) are exactly those, for whose group algebras L.Paris in [9] presented examples of multiplicative  $K$ -bases.

**2. Results.** Our main results are the following:

**Theorem 1.** *Let  $K$  be a field of characteristic  $p$  and let  $G$  satisfy one of the following conditions:*

1.  $G$  is a powerful  $p$ -group;
2.  $p$  is odd,  $G$  is a 2-generated  $p$ -group with the central cyclic commutator subgroup.

*Then the group algebra  $KG$  does not have a filtered multiplicative  $K$ -basis.*

**Theorem 2.** *Let  $K$  be a field of characteristic  $p$  and let  $G$  be a nonabelian  $p$ -group with a cyclic subgroup of index  $p^2$ . Then the group algebra  $KG$  possesses a filtered multiplicative  $K$ -basis if and only if  $p = 2$  and one of the following conditions is satisfied:*

1.  $G$  is either the dihedral 2-group or  $D_{2^m} \times C_2$  or the central product  $D_8 \times C_4$  of  $D_8$  with  $C_4$ ;
2.  $K$  contains a primitive cube root of the unity and  $G$  is either  $Q_8 \times C_2$  or  $Q_8$ ;
3.  $G$  is one of the following groups:

$$G_5 = \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, \\ (d, a) = (d, c) = 1, (c, a) = d \rangle, \text{ with } m \geq 4;$$

$$G_{13} = \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, \\ (d, a) = (d, c) = 1, (c, a) = a^2 d \rangle, \text{ with } m \geq 5;$$

$$G_{14} = \langle a, c, d \mid a^{2^{m-2}} = d^2 = 1, c^2 = a^{2^{m-3}}, \\ (d, a) = (d, c) = 1, (c, a) = a^2 d \rangle, \text{ with } m \geq 5;$$

$$\begin{aligned}
G_{17} &= \langle a, c, d \mid a^{2^{m-2}} = d^2 = c^2 = 1, \\
&\quad (d, a) = a^{2^{m-3}}, (d, c) = 1, (c, a) = d \rangle, \text{ with } m \geq 5; \\
G_{18} &= \langle a, c, d \mid a^{2^{m-2}} = d^2 = 1, c^2 = d, \\
&\quad (d, a) = a^{2^{m-3}}, (c, a) = a^2 d \rangle, \text{ with } m \geq 4; \\
G_{22} &= \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, (d, a) = 1, \\
&\quad (d, c) = a^{2^{m-3}}, (c, a) = a^{-2^{m-4}} d \rangle, \text{ with } m \geq 6; \\
G_{23} &= \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, (d, a) = 1, \\
&\quad (d, c) = a^{2^{m-3}}, (c, a) = a^{2-2^{m-4}} d \rangle, \text{ with } m \geq 6; \\
G_{24} &= \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, (d, c) = 1, \\
&\quad (d, a) = a^{2^{m-3}}, (c, a) = a^{2-2^{m-4}} d \rangle, \text{ with } m \geq 6; \\
G_{25} &= \langle a, c, d \mid a^{2^{m-2}} = d^2 = 1, c^2 = a^{2^{m-3}}, (d, c) = 1, \\
&\quad (d, a) = a^{2^{m-3}}, (c, a) = a^{2-2^{m-4}} d \rangle, \text{ with } m \geq 5.
\end{aligned}$$

**3. Preliminary remarks and notations.** Let  $B$  be a filtered multiplicative  $K$ -basis in a finite-dimensional  $K$ -algebra  $A$ . In the proof of the main result we use the following simple properties of  $B$  (see [2]):

- (I)  $B \cap \text{rad}(A)^n$  is a  $K$ -basis of  $\text{rad}(A)^n$  for all  $n \geq 1$ .
- (II) If  $u, v \in B \setminus \text{rad}(A)^k$  and  $u \equiv v \pmod{\text{rad}(A)^k}$  then  $u = v$ .

Recall that the *Frattini subalgebra*  $\Phi(A)$  of  $A$  is defined as the intersection of all maximal subalgebras of  $A$  if those exist, and as  $A$  otherwise. In [3] it was shown that if  $A$  is a nilpotent algebra over a field  $K$ , then  $\Phi(A) = A^2$ . It follows that

- (III) If  $B$  is a filtered multiplicative  $K$ -basis of  $A$  and if  $B \setminus \{1\} \subseteq \text{rad}(A)$ , then all elements of  $B \setminus \text{rad}(A)^2$  are generators of  $A$  over  $K$ .

Let  $K$  be a field of characteristic  $p$  and  $G$  be a finite  $p$ -group. For  $a, b \in G$  we define  $a^b = b^{-1}ab$  and  $(a, b) = a^{-1}b^{-1}ab$ . Let  $Q_{2^n}$ ,  $D_{2^n}$  and  $C_{p^n}$  be the *generalized quaternion group*, the *dihedral 2-group* of order  $2^n$ , and the *cyclic group* of order  $p^n$ , respectively.

A  $p$ -group  $G$  is called *powerful*, if  $p = 2$  then  $G/G^4$  is abelian, or if  $p > 2$  and  $G/G^p$  is abelian.

The ideal  $I_K(G) = \{ \sum_{g \in G} \alpha_g g \in KG \mid \sum_{g \in G} \alpha_g = 0 \}$  is called the *augmentation ideal* of  $KG$  and

$$I_K(G) \supset I_K^2(G) \supset \cdots \supset I_K^s(G) \supset I_K^{s+1}(G) = 0.$$

Then the subgroup  $\mathfrak{D}_n(G) = \{ g \in G \mid g - 1 \in I_K^n(G) \}$  is called the  *$n$ th dimensional subgroup* of  $KG$ .

We define the *Lazard-Jennings series* ( see [4] )

$$M_1(G) \supseteq M_2(G) \supseteq \cdots \supseteq M_t(G) = 1$$

as follows: Put  $M_1(G) = G$  and  $M_i(G) = \langle (M_{i-1}(G), G), M_{[\frac{i}{p}]}^p(G) \rangle$ , where

- $[\frac{i}{p}]$  is the smallest integer not less than  $\frac{i}{p}$ ;
- $(M_{i-1}(G), G) = \langle (u, v) \mid u \in M_i(G), v \in G \rangle$ ;
- $M_i^p(G)$  is the subgroup generated by  $p$ -powers of the elements of  $M_i(G)$ .

We know that for the finite  $p$ -groups  $M_i(G) = \mathfrak{D}_i(G)$  for all  $i$ .

Let  $\mathbb{I} = \{i \in \mathbb{N} \mid \mathfrak{D}_i(G) \neq \mathfrak{D}_{i+1}(G)\}$  and let  $p^{d_i}$  ( $i \in \mathbb{I}$ ) be the order of the elementary abelian  $p$ -group

$$\mathfrak{D}_i(G)/\mathfrak{D}_{i+1}(G) = \otimes_{j=1}^{d_i} \langle u_{ij} \mathfrak{D}_{i+1}(G) \rangle.$$

Any element  $g \in G$  can be written uniquely as

$$g = u_{11}^{\alpha_{11}} u_{12}^{\alpha_{12}} \cdots u_{1d_1}^{\alpha_{1d_1}} u_{21}^{\alpha_{21}} \cdots u_{2d_2}^{\alpha_{2d_2}} \cdots u_{i1}^{\alpha_{i1}} \cdots u_{id_i}^{\alpha_{id_i}} \cdots u_{s1}^{\alpha_{s1}} \cdots u_{sd_s}^{\alpha_{sd_s}},$$

where  $i \in \mathbb{I}$ ,  $u_{ij} \in \mathfrak{D}_i(G)$ ,  $0 \leq \alpha_{ij} < p$ , and  $s$  is defined as above.

Elements of the form  $w = \prod_{l \in \mathbb{I}} (\prod_{k=1}^{d_l} (u_{lk} - 1)^{y_{lk}}) \in I_K(G)$ , where indices of its factors are in lexicographic order and  $0 \leq y_{lk} < p$ , are called regular elements. Its elements have weight  $\mu(w) = \sum_{l \in \mathbb{I}} (\sum_{k=1}^{d_l} l y_{lk})$ . By Jennings Theorem ( see [4] ), the regular elements of weight greater than or equal to  $t$  constitute a  $K$ -basis for the ideal  $I_K^t(G)$ . Since  $\mathfrak{D}_2(G) = \Phi(G)$ , where  $\Phi(G)$  is the Frattini subgroup of  $G$ , we have that the set  $\{u_{11}, u_{12}, \dots, u_{1d_1}\}$  is a minimal generator system of  $G$ .

Clearly,  $I_K(G)$  is the radical of  $KG$ . Suppose that  $B_1 = \{1, b_1, \dots, b_{|G|-1}\}$  is a filtered multiplicative  $K$ -basis of  $KG$ . Then  $B = B_1 \setminus \{1\}$  is a filtered multiplicative  $K$ -basis of  $I_K(G)$  and consists of  $|G| - 1$  elements. By Jennings Theorem in [4]  $\{ (u_{1j} - 1) + I_K^2(G) \mid j = 1, \dots, d_1 \}$  form a  $K$ -basis of  $I_K(G)/I_K^2(G)$ .

Put  $n = d_1$  and  $B \setminus (B \cap I_K^2(G)) = \{b_1, b_2, \dots, b_n\}$ . Thus

$$b_k \equiv \sum_{i=1}^n \alpha_{ki} (u_{1i} - 1) \pmod{I_K^2(G)},$$

where  $\alpha_{ki} \in K$  and  $\Delta = \det(\alpha_{ki}) \neq 0$ .

Clearly,  $z_{ji} = (u_{1j}, u_{1i}) \in \mathfrak{D}_2(G)$  and  $z_{ji} - 1 \in I_K^2(G)$ . Using the identity

$$\begin{aligned} (y-1)(x-1) &= [(x-1)(y-1) + (x-1) + (y-1)](z-1) \\ &\quad + (x-1)(y-1) + (z-1), \end{aligned} \tag{1}$$

where  $z = (y, x)$ , we obtain that

$$(u_{1j} - 1)(u_{1i} - 1) = (u_{1i} - 1)(u_{1j} - 1) + (z_{ji} - 1) \pmod{I_K^3(G)}. \quad (2)$$

Then

$$\begin{aligned} b_k b_s \equiv & \sum_{i=1}^n \alpha_{ki} \alpha_{si} (u_{1i} - 1)^2 + \sum_{\substack{i,j=1 \\ i < j}}^n (\alpha_{ki} \alpha_{sj} + \alpha_{kj} \alpha_{si}) (u_{1i} - 1)(u_{1j} - 1) \\ & + \sum_{\substack{i,j=1 \\ i < j}}^n \alpha_{kj} \alpha_{si} (z_{ji} - 1) \pmod{I_K^3(G)}, \end{aligned} \quad (3)$$

where  $k, s = 1, \dots, n$ .

Let us compute the dimension subgroups. If  $p \neq 2$  then:

$$\mathfrak{D}_1(G) = G, \quad \mathfrak{D}_2(G) = \Phi(G), \quad \mathfrak{D}_3(G) = \langle (\mathfrak{D}_2(G), G), G^p \rangle,$$

$$\mathfrak{D}_4(G) = \langle (\mathfrak{D}_3(G), G), G^p \rangle, \dots, \mathfrak{D}_p(G) = \langle (\mathfrak{D}_{p-1}(G), G), G^p \rangle,$$

and if  $p = 2$ , we get the following:

$$\mathfrak{D}_1(G) = G, \quad \mathfrak{D}_2(G) = \Phi(G) = G^2, \quad \mathfrak{D}_3(G) = \langle (\mathfrak{D}_2(G), G), G^4 \rangle.$$

Assume that  $G$  is a powerful  $p$ -group, i.e. if  $p = 2$  then  $G' < G^4$ , and  $G' < G^p$  for  $p > 2$ . Then  $z_{ji} \in \mathfrak{D}_3(G)$  and  $z_{ji} - 1 \in I_K^3(G)$ . By (2) it follows that  $b_k b_s \equiv b_s b_k \pmod{I_K^3(G)}$ .

Let  $b_k b_s \in I_K^3(G)$ . Since the elements

$$\{ (u_{1i} - 1)^2; (u_{1k} - 1)(u_{1l} - 1); (u_{2j} - 1) \mid i, k, l = 1, \dots, n; k < l; j = 1, \dots, d_2 \}$$

have weight 2, by Jennings Theorem ( see [4] ), these elements form a basis of  $I_K^2(G)$  modulo  $I_K^3(G)$ . Because of (3) we have that  $\alpha_{ki} \alpha_{si} = 0$  and  $\alpha_{ki} \alpha_{sj} + \alpha_{si} \alpha_{kj} = 0$ , where  $i, j = 1, \dots, n$ . Therefore, all minors of order two, which are formed from the  $k$  and  $s$  lines of the matrix  $(\alpha_{i,j})$ , equal zero. From this follows that the determinant of the matrix  $(\alpha_{i,j})$  is zero, which is impossible.

Therefore,  $b_k b_s, b_s b_k \notin I_K^3(G)$  and  $b_k b_s \equiv b_s b_k \pmod{I_K^3(G)}$  and by property (II) of the filtered multiplicative  $K$ -basis we conclude that  $b_k b_s = b_s b_k$  (for all  $k, s = 1, \dots, n$ ) and  $I_K(G)$  is a commutative algebra, which is a contradiction.

Let  $K$  be a field of characteristic  $p$ , where  $p$  is odd, and let  $G$  be a 2-generated  $p$ -group with a central cyclic commutator subgroup. Then by Theorem 1 in [7]

$$\begin{aligned} G = \langle a, c \mid d = (c, a), a^{p^{m_1}} = d^{R p^r}, c^{p^{m_2}} = d^{S p^s}, \\ d^{p^{m_3}} = 1, (d, a) = (d, c) = 1 \rangle, \end{aligned}$$

where  $m_1, m_2, m_3, R, r, S, s$  are natural numbers defined in [7].

Clearly,  $d = (c, a) \in \mathfrak{D}_2(G)$  and  $d - 1 \in I_K^2(G)$ . Then

$$(c - 1)(a - 1) \equiv (a - 1)(c - 1) + (d - 1) \pmod{I_K^3(G)},$$

$$(c - 1)^2(a - 1) \equiv (a - 1)(c - 1)^2 + 2(c - 1)(d - 1) \pmod{I_K^4(G)}.$$

Put

$$\begin{cases} b_1 \equiv \alpha_1(a - 1) + \alpha_2(c - 1) \pmod{I_K^2(G)}; \\ b_2 \equiv \beta_1(a - 1) + \beta_2(c - 1) \pmod{I_K^2(G)}, \end{cases} \quad (4)$$

where  $\alpha_i, \beta_i \in K$  and  $\Delta = \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ .

In the rest of the proof we can assume that  $d - 1 \notin I_K^3(G)$ . In the opposite case, as we have shown above, we have a contradiction. Thus  $d - 1$  has weight 2.

Now let us compute  $b_{i_1}b_{i_2}b_{i_3}$  modulo  $I_K^4(G)$ , ( $i_k = 1, 2$ ). The result of our computation will be written in a table, consisting of the coefficients of the decomposition  $b_{i_1}b_{i_2}b_{i_3}$  with respect to the basis

$$\{ (a - 1)^{j_1}(c - 1)^{j_2}(d - 1)^{j_3} \mid j_1 + j_2 + 2j_3 = 3; j_1, j_2 = 0, \dots, 3; j_3 = 0, 1 \}$$

of the ideal  $I_K^3(G)$ . We will divide the table into two parts (the second part written below the first part). The coefficients corresponding to the first three basis elements will be in the first part of the table, while the next three will be in the second one.

	$(a - 1)^3$	$(c - 1)^3$	$(a - 1)^2(c - 1)$
$b_1b_2b_1$	$\alpha_1^2\beta_1$	$\alpha_2^2\beta_2$	$\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1$
$b_1b_2^2$	$\alpha_1\beta_1^2$	$\alpha_2\beta_2^2$	$\alpha_2\beta_1^2 + 2\alpha_1\beta_1\beta_2$
$b_2^2b_1$	$\alpha_1\beta_1^2$	$\alpha_2\beta_2^2$	$\alpha_2\beta_1^2 + 2\alpha_1\beta_1\beta_2$
$b_1^2b_2$	$\alpha_1^2\beta_1$	$\alpha_2^2\beta_2$	$\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1$
$b_2b_1^2$	$\alpha_1^2\beta_1$	$\alpha_2^2\beta_2$	$\alpha_1^2\beta_2 + 2\alpha_1\alpha_2\beta_1$
$b_2b_1b_2$	$\alpha_1\beta_1^2$	$\alpha_2\beta_2^2$	$\alpha_2\beta_1^2 + 2\alpha_1\beta_1\beta_2$
$b_1^3$	$\alpha_1^3$	$\alpha_2^3$	$3\alpha_1^2\alpha_2$
$b_2^3$	$\beta_1^3$	$\beta_2^3$	$3\beta_1^2\beta_2$

	$(a - 1)(c - 1)^2$	$(a - 1)(d - 1)$	$(c - 1)(d - 1)$
$b_1b_2b_1$	$\alpha_2^2\beta_1 + 2\alpha_1\alpha_2\beta_2$	$\alpha_1^2\beta_2$	$\alpha_2^2\beta_1 + 2\alpha_1\alpha_2\beta_2$
$b_1b_2^2$	$\alpha_1\beta_2^2 + 2\alpha_2\beta_1\beta_2$	$\alpha_1\beta_1\beta_2$	$3\alpha_2\beta_1\beta_2$
$b_2^2b_1$	$\alpha_1\beta_2^2 + 2\alpha_2\beta_1\beta_2$	$3\alpha_1\beta_1\beta_2$	$\alpha_2\beta_1\beta_2 + 2\alpha_1\beta_2^2$
$b_1^2b_2$	$\alpha_2^2\beta_1 + 2\alpha_1\alpha_2\beta_2$	$3\alpha_1\alpha_2\beta_1$	$2\alpha_2^2\beta_1 + \alpha_1\alpha_2\beta_2$
$b_2b_1^2$	$\alpha_2^2\beta_1 + 2\alpha_1\alpha_2\beta_2$	$\alpha_1\alpha_2\beta_1$	$3\alpha_1\alpha_2\beta_2$
$b_2b_1b_2$	$\alpha_1\beta_2^2 + 2\alpha_2\beta_1\beta_2$	$\alpha_2\beta_1^2$	$\alpha_1\beta_2^2 + 2\alpha_2\beta_1\beta_2$
$b_1^3$	$3\alpha_1\alpha_2^2$	$3\alpha_1^2\alpha_2$	$3\alpha_1\alpha_2^2$
$b_2^3$	$3\beta_1\beta_2^2$	$3\beta_1^2\beta_2$	$3\beta_1\beta_2^2$

We have obtained 8 elements. If  $\text{char}K > 3$  or  $\text{char}K = 3$  and  $|a| \neq 3 \neq |c|$ , then the  $K$ -dimension of  $I_K^3(G)/I_K^4(G)$  equals 6.

In the opposite case the  $K$ -dimension of  $I_K^3(G)/I_K^4(G)$  is less than 6. But we must have either 6 or less, respectively, linearly independent elements  $b_{i_1}b_{i_2}b_{i_3}$  modulo the ideal  $I_K^4(G)$ . Therefore, some of these elements either are equal to zero modulo the ideal  $I_K^4(G)$  or coincide with some other elements of the system. Then we get by simple calculations that  $\Delta = 0$ , which is impossible for  $\text{char}K > 2$ .

Indeed, for example, if  $b_1b_2b_1 \equiv 0 \pmod{I_K^4(G)}$  then from the first two columns follows that either  $\alpha_2 = \beta_1 = 0$  or  $\alpha_1 = \beta_2 = 0$ . Also, from the 3th and 6th columns we get either  $\alpha_1^2\beta_2 = 0$  or  $\alpha_2^2\beta_1 = 0$ , respectively. Therefore,  $\Delta = 0$ , which is impossible.

Now, for example, put  $\beta_1\beta_2\beta_1 \equiv \beta_1\beta_2^2 \pmod{I_K^4(G)}$ . It follows that  $\alpha_1 = \beta_1$ ,  $\alpha_2 = \beta_2$  and  $\Delta = 0$ , which is also a contradiction.

The rest of the cases are similar to these two.

**4. Proof of Theorem 2.** According to Theorems 1 and 2 in [8] there are 11 finite nonabelian  $p$ -groups of order  $p^m$  and exponent  $p^{m-2}$  for  $p > 2$  and 26 such groups when  $p = 2$ .

In the rest of the proof we will keep the indexes of these groups as in [8], but to make the calculation easier we will rename the generators of these groups as follows: we do not change  $a$  and  $c$  but we will denote  $d$  either as  $b$  or  $b^{-1}$ .

First, we consider the case when  $\text{char}K > 2$ . Then by Theorem 1 in [8] a finite nonabelian neither metacyclic nor powerful  $p$ -group of order  $p^m$  and exponent  $p^{m-2}$  is isomorphic to one of the groups  $G_1, G_5, G_6, G_7, G_{11}$ .

$$\begin{aligned} \text{If } G = G_1 = \langle a, c, d \mid a^{p^{m-2}} = c^p = d^p = 1, \\ (d, a) = 1, (d, c) = 1, (c, a) = d \rangle, \text{ with } m \geq 3, \end{aligned}$$

then by Theorem 1 we have that  $KG$  has no filtered multiplicative  $K$ -basis.

Let  $G$  be one of the following groups:

$$\begin{aligned} H(r) &= \langle a, c, d \mid a^{p^{m-2}} = c^p = d^p = 1, (d, a) = 1, \\ &\quad (c, a) = d, (d, c) = a^{-rp^{m-3}} \rangle, \text{ with } m \geq 4; \\ G_7 &= \langle a, c, d \mid a^{p^{m-2}} = c^p = d^p = 1, \\ &\quad (d, a) = a^{p^{m-3}}, (c, a) = d, (d, c) = 1 \rangle, \text{ with } m \geq 4; \\ G_{11} &= \langle a, c, d \mid a^9 = d^3 = 1, \\ &\quad a^3 = c^3, (d, a) = 1, (c, a) = d, (d, c) = a^3 \rangle, \end{aligned}$$

where either  $r = 1$  or  $r$  is a quadratic nonresidue modulo  $p$ . Note that if  $r = 1$  then  $H(1)$  coincides with  $G_5$  and in the other case  $H(r)$  coincides with  $G_6$ .

Then by (1) we get

$$(c-1)(a-1) \equiv (a-1)(c-1) + (d-1) \pmod{I_K^3(G)}.$$

Now, similarly to the proof of Theorem 1 above, let us compute  $b_{i_1}b_{i_2}b_{i_3}$  modulo  $I_K^4(G)$ , ( $i_k = 1, 2$ ). The result of our computation can be written in a similar table, consisting of the coefficients of the decomposition  $b_{i_1}b_{i_2}b_{i_3}$  with respect to the considered basis

$$\{ (a-1)^3, (a-1)^2(c-1), (a-1)(d-1), (c-1)(d-1), (a-1)(c-1)^2 \}$$

of the ideal  $I_K^3(G)$ .

We must consider the following three cases.

Case 1. Let either  $G \neq G_7$  with  $p = 3$  and  $m = 4$  or  $G \neq H(r)$  with  $p = 3$ ,  $m = 4$  or  $G \neq G_{11}$ . Then by (1), the element  $d-1$  is central modulo  $I_K^4(G)$ , and we will have the same table as in the proof of Theorem 1. Thus, we will have a contradiction.

Case 2. Let  $G = G_7$  with  $p = 3$  and  $m = 4$ . Then

$$G = \langle a, c, d \mid a^9 = c^3 = d^3 = 1, (d, a) = a^3, (c, a) = d, (d, c) = 1 \rangle$$

and by (1) we get  $(d-1)(c-1) \equiv (c-1)(d-1) \pmod{I_K^4(G)}$  and

$$(d-1)(a-1) \equiv (a-1)(d-1) + (a-1)^3 \pmod{I_K^4(G)}.$$

Modulo  $I_K^4(G)$  it follows that

	$(a-1)^3$	$(a-1)^2(c-1)$	$(a-1)(d-1)$	$(c-1)(d-1)$	$(a-1)(c-1)^2$
$b_1b_2b_1$	$\alpha_1\beta_1(\alpha_1 + \alpha_2)$	$\alpha_1\Delta$	$\alpha_1^2\beta_2$	$-\alpha_2\Delta$	$-\alpha_2\Delta$
$b_1b_2^2$	$\beta_1^2(\alpha_1 + \alpha_2)$	$-\beta_1\Delta$	$\alpha_1\beta_1\beta_2$	0	$\beta_2\Delta$
$b_2b_1^2$	$\alpha_1\beta_1(\beta_1 + \beta_2)$	$\alpha_1\Delta$	$\alpha_1\alpha_2\beta_1$	0	$-\alpha_2\Delta$
$b_2b_1b_2$	$\alpha_1\beta_1(\beta_1 + \beta_2)$	$\beta_1\Delta$	$\alpha_2\beta_1^2$	$\beta_2\Delta$	$\beta_2\Delta$
$b_1^3$	$\alpha_1^2(\alpha_1 + \alpha_2)$	0	$\alpha_1^2\alpha_2$	0	0
$b_1^2b_2$	$\alpha_1\beta_1(\alpha_1 + \alpha_2)$	$\alpha_1\Delta$	$\alpha_1\alpha_2\beta_1$	$\alpha_2\Delta$	$-\alpha_2\Delta$
$b_2^2b_1$	$\alpha_1\beta_1(\beta_1 + \beta_2)$	$-\beta_1\Delta$	$\alpha_1\alpha_1\beta_2$	$-\beta_2\Delta$	$\beta_2\Delta$
$b_2^3$	$\beta_1^2(\beta_1 + \beta_2)$	0	$\beta_1^2\beta_2$	0	0

We have obtained 8 elements, but the  $K$ -dimension of  $I_K^3(G)/I_K^4(G)$  equals 5. Therefore, some of these elements either are equal to zero modulo the ideal  $I_K^4(G)$  or coincide with some other elements of the system. Then we get by simple calculations that  $\Delta = 0$ , which is impossible.

Case 3. Let either  $G = H(r)$  with  $p = 3$ ,  $m = 4$  or  $G = G_{11}$ . Then by (1) we get

$$(d-1)(c-1) \equiv (c-1)(d-1) + (a-1)^3 \pmod{I_K^4(G)},$$

$$(d-1)(a-1) \equiv (a-1)(d-1) \pmod{I_K^4(G)}.$$



Modulo  $I_K^4(G)$  it follows that

	$(a-1)^3$	$(a-1)^2(c-1)$	$(a-1)(d-1)$	$(a-1)(c-1)^2$	$(c-1)(d-1)$
$b_1 b_2 b_1$	$\beta_1(\alpha_1^2 + \alpha_2^2)$	$\alpha_1 \Delta$	$\alpha_1 \Delta$	$-\alpha_2 \Delta$	$-\alpha_2 \Delta$
$b_1 b_2^2$	$\beta_1(\alpha_1 \beta_1 + \alpha_2 \beta_2)$	$-\beta_1 \Delta$	$\beta_1 \Delta$	$\beta_2 \Delta$	0
$b_2 b_1^2$	$\alpha_1(\alpha_1 \beta_1 + \alpha_2 \beta_2)$	$\alpha_1 \Delta$	$-\alpha_1 \Delta$	$-\alpha_2 \Delta$	0
$b_2 b_1 b_2$	$\alpha_1(\beta_1^2 + \beta_2^2)$	$-\beta_1 \Delta$	$-\beta_1 \Delta$	$\beta_2 \Delta$	$\beta_2 \Delta$
$b_1^3$	$\alpha_1(\alpha_1^2 + \alpha_2^2)$	0	0	0	0
$b_1^2 b_2$	$\alpha_1(\alpha_1 \beta_1 + \alpha_2 \beta_2)$	$\alpha_1 \Delta$	0	$-\alpha_2 \Delta$	$\alpha_2 \Delta$
$b_2^2 b_1$	$\beta_1(\alpha_1 \beta_1 + \alpha_2 \beta_2)$	$-\beta_1 \Delta$	0	$\beta_2 \Delta$	$-\beta_2 \Delta$
$b_2^3$	$\beta_1(\beta_1^2 + \beta_2^2)$	0	0	0	0

Now, similarly to case 2, we have obtained 8 elements, but the  $K$ -dimension of  $I_K^3(G)/I_K^4(G)$  equals 5. Therefore, some of these elements either identically equal zero modulo the ideal  $I_K^4(G)$  or coincide with some other elements of the system. Then we get by simple calculations that  $\Delta = 0$ , which is impossible.

Therefore, if  $p$  is odd, then  $KG$  has no filtered multiplicative  $K$ -basis.

Now let  $\text{char} K = 2$ . Then by Theorem 2 in [8] a finite nonabelian neither metacyclic nor powerful 2-group of order  $2^m$  and exponent  $2^{m-2}$  is isomorphic to one of the groups  $G_2, G_3, G_5, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, G_{22}, G_{23}, G_{24}, G_{25}, G_{26}$  with the exception of  $G_4$ , which is discussed below. If  $m = 5$  then  $G_{25}$  coincides with  $G_{26}$ .

First, let us suppose that  $G$  is

$$G_4 = \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, \\ (d, a) = (c, a) = 1, (d, c) = a^{2^{m-3}} \rangle, \text{ with } m \geq 4.$$

Clearly, for  $m \geq 5$ ,  $G$  is powerful, therefore  $G_4$  is the central product of  $D_8 = \langle a, b \rangle$  and  $C_4 = \langle c \rangle$ . Put  $b_1 = (1 + a) + (1 + c)$ ,  $b_2 = 1 + c$  and  $b_3 = 1 + d$ . Thus  $\{ b_2^i b_1^j b_2^k b_3^l \mid i, k = 0, 1; j, l = 0, \dots, 3 \}$  form a filtered multiplicative  $K$ -basis of  $KG_4$ .

Let  $G$  be either  $G_2 = Q_{2^{n-1}} \times C_2$  or  $G_3 = D_{2^{m-1}} \times C_2$ . Then using [2], one obtains that the group  $G$  satisfies conditions 1. or 2. of Theorem 2.

Let  $G$  be one of the following groups:

$$G_{11} = \langle a, c, d \mid a^{2^{m-1}} = c^2 = d^2 = 1, (d, c) = 1, \\ (d, a) = 1, (c, a) = a^{2+2^{m-2}} \rangle, \text{ with } m \geq 4; \\ G_{12} = \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, (d, c) = a^{2^{m-3}}, \\ (d, a) = 1, (c, a) = a^2 \rangle, \text{ with } m \geq 5;$$

$$\begin{aligned}
G_{15} &= \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, (d, c) = 1, \\
&\quad (d, a) = a^{2^{m-3}}, (c, a) = a^{2+2^{m-3}} \rangle, \text{ with } m \geq 5; \\
G_{16} &= \langle a, c, d \mid a^{2^{m-2}} = c^2 = d^2 = 1, (d, c) = a^{2^{m-3}}, \\
&\quad (d, a) = a^{2^{m-3}}, (c, a) = a^{2+2^{m-3}} \rangle, \text{ with } m \geq 5.
\end{aligned}$$

It is easy to see that  $G$  is 3-generated and we can put

$$\begin{cases} b_1 \equiv \alpha_1(1+a) + \alpha_2(1+c) + \alpha_3(1+d) \pmod{I_K^2(G)}; \\ b_2 \equiv \beta_1(1+a) + \beta_2(1+c) + \beta_3(1+d) \pmod{I_K^2(G)}; \\ b_3 \equiv \gamma_1(1+a) + \gamma_2(1+c) + \gamma_3(1+d) \pmod{I_K^2(G)}, \end{cases}$$

where  $\alpha_i, \beta_i, \gamma_i \in K$  and  $\Delta \neq 0$ .

By (1) we have that  $(1+c)(1+a) \equiv (1+a)(1+c) + (1+a)^2 \pmod{I_K^3(G)}$  and  $1+d$  is central by modulo  $I_K^3(G)$ .

Now let us compute  $b_i b_j$  by modulo  $I_K^3(G)$ ,  $(i, j = 1, 2)$ . The result of our computation will be written in a table, consisting of the coefficients of the decomposition  $b_i b_j$  with respect to the basis  $\{ (1+a)^2, (1+a)(1+c), (1+a)(1+d), (1+c)(1+d) \}$  of the ideal  $I_K^2(G)$ .

	$(1 + a)^2$	$(1 + a)(1 + c)$	$(1 + a)(1 + d)$	$(1 + c)(1 + d)$
$b_1 b_2$	$\beta_1(\alpha_1 + \alpha_2)$	$\alpha_1 \beta_2 + \alpha_2 \beta_1$	$\alpha_1 \beta_3 + \alpha_3 \beta_1$	$\alpha_2 \beta_3 + \alpha_3 \beta_2$
$b_2 b_1$	$\alpha_1(\beta_1 + \beta_2)$	$\alpha_1 \beta_2 + \alpha_2 \beta_1$	$\alpha_1 \beta_3 + \alpha_3 \beta_1$	$\alpha_2 \beta_3 + \alpha_3 \beta_2$
$b_1 b_3$	$\gamma_1(\alpha_1 + \alpha_2)$	$\alpha_1 \gamma_2 + \alpha_2 \gamma_1$	$\alpha_1 \gamma_3 + \alpha_3 \gamma_1$	$\alpha_2 \gamma_3 + \alpha_3 \gamma_2$
$b_3 b_1$	$\alpha_1(\gamma_1 + \gamma_2)$	$\alpha_1 \gamma_2 + \alpha_2 \gamma_1$	$\alpha_1 \gamma_3 + \alpha_3 \gamma_1$	$\alpha_2 \gamma_3 + \alpha_3 \gamma_2$
$b_2 b_3$	$\gamma_1(\beta_1 + \beta_2)$	$\beta_1 \gamma_2 + \beta_2 \gamma_1$	$\beta_1 \gamma_3 + \beta_3 \gamma_1$	$\beta_2 \gamma_3 + \beta_3 \gamma_2$
$b_3 b_2$	$\beta_1(\gamma_1 + \gamma_2)$	$\beta_1 \gamma_2 + \beta_2 \gamma_1$	$\beta_1 \gamma_3 + \beta_3 \gamma_1$	$\beta_2 \gamma_3 + \beta_3 \gamma_2$
$b_1^2$	$\alpha_1(\alpha_1 + \alpha_2)$	0	0	0
$b_2^2$	$\beta_1(\beta_1 + \beta_2)$	0	0	0
$b_3^2$	$\gamma_1(\gamma_1 + \gamma_2)$	0	0	0

We have obtained 9 elements. But the  $K$ -dimension of  $I_K^2(G)/I_K^3(G)$  equals 4 and we must have four linearly independent elements  $b_i b_j$  modulo ideal  $I_K^3(G)$ . Therefore, some of these elements either are equal to zero modulo the ideal  $I_K^3(G)$  or coincide with some other elements of the system.

We will consider the following cases:

Case 1. Let  $b_i b_j \equiv b_j b_i \pmod{I_K^3(G)}$ . But  $b_i b_j, b_j b_i \notin I_K^3(G)$  and by property (II) we conclude that  $I_K(G)$  is a commutative algebra, which is a contradiction.

Case 2. It is easy to see that the first six lines equal neither zero nor the last three lines, because from the last three columns in our table follows that all minors equal zero and  $\Delta = 0$ , which is impossible.

Case 3. Let  $b_i^2 \equiv 0 \pmod{I_K^3(G)}$  for two values  $i$ , for example,  $b_1^2 \equiv b_2^2 \equiv 0 \pmod{I_K^3(G)}$ . Then either  $\alpha_1 = 0$  or  $\alpha_1 = \alpha_2$  and either  $\beta_1 = 0$  or  $\beta_1 = \beta_2$ , respectively. Since we must have 4 linearly independent elements and the cases 1 and 2 are impossible, we have that  $b_3^2 \equiv 0 \pmod{I_K^3(G)}$ . Thus either  $\gamma_1 = 0$  or  $\gamma_1 = \gamma_2$ . If we put the values of  $\alpha_j, \beta_j$  and  $\gamma_j$  into our table ( we will have eight cases) then we will get a contradiction.

Therefore, in any other case, we can refer back to these three cases.

Suppose  $G$  is one of the groups  $G_5, G_{17}, G_{22}$  or  $G_{25}$  with  $m = 5$  ( i.e.  $G_{26}$  ) from Theorem 2. Clearly,  $G$  is two-generated and we can assume that  $u = b_1, v = b_2$ , where  $b_1, b_2$  can be written as in (4).

Then by (1) we get  $1 + d$  is central modulo  $I_K^3(G)$  and

$$(1 + c)(1 + a) \equiv (1 + a)(1 + c) + (1 + d) \pmod{I_K^3(G)}.$$

It follows that

$$\begin{cases} uv \equiv \alpha_1\beta_1(1 + a)^2 + \Delta(1 + a)(1 + c) + \alpha_2\beta_1(1 + d) & \pmod{I_K^3(G)}; \\ vu \equiv \alpha_1\beta_1(1 + a)^2 + \Delta(1 + a)(1 + c) + \alpha_1\beta_2(1 + d) & \pmod{I_K^3(G)}; \\ u^2 \equiv \alpha_1^2(1 + a)^2 + \alpha_1\alpha_2(1 + d) & \pmod{I_K^3(G)}; \\ v^2 \equiv \beta_1^2(1 + a)^2 + \beta_1\beta_2(1 + d) & \pmod{I_K^3(G)}. \end{cases}$$

Since the  $K$ -dimension of  $I_K^2(G)/I_K^3(G)$  equals 3 and  $\Delta \neq 0$ , we have that  $uv \neq vu \pmod{I_K^3(G)}$  and  $u^2 \neq v^2 \pmod{I_K^3(G)}$ . Thus either  $v^2 \equiv 0 \pmod{I_K^3(G)}$  and  $\beta_1 = 0$  or  $u^2 \equiv 0 \pmod{I_K^3(G)}$  and  $\alpha_1 = 0$ . It is easy to see that the second case is symmetric to the first one, so we consider only the first one. Therefore  $\alpha_1 \neq 0$  and we can put  $u = (1 + a) + \mu(1 + c)$  and  $v = 1 + c$ , where  $\mu = \frac{\alpha_2}{\alpha_1}$ .

We will prove that the elements  $\{u^i, vu^{i-1}, vuvu^{i-3}, uvu^{i-2}\}$  form a basis of  $I_K^i(G)/I_K^{i+1}(G)$ , where  $i > 3$ .

First of all,  $vuv \equiv (1 + c)(1 + d) \pmod{I_K^4(G)}$ , and by induction we get that

$$\begin{aligned} u^{4i} &\equiv (1 + a)^{4i} \pmod{I_K^{4i+1}(G)}, \\ u^{4i+1} &\equiv (1 + a)^{4i+1} + \mu(1 + a)^{4i}(1 + c) \pmod{I_K^{4i+2}(G)}, \\ u^{4i+2} &\equiv (1 + a)^{4i+2} + \mu(1 + a)^{4i}(1 + d) \pmod{I_K^{4i+3}(G)}, \\ u^{4i+3} &\equiv (1 + a)^{4i+3} + \mu(1 + a)^{4i+2}(1 + c) \\ &\quad + \mu(1 + a)^{4i+1}(1 + d) + \mu^2(1 + a)^{4i}(1 + c)(1 + d) \pmod{I_K^{4i+4}(G)}. \end{aligned}$$

We consider the following four cases:

Case 1. If  $i \equiv 0 \pmod{4}$ , then we have

$$\begin{aligned} u^i &\equiv (1+a)^i \pmod{I_K^{i+1}(G)}, \\ vu^{i-1} &\equiv (1+a)^{i-1}(1+c) + (1+a)^{i-2}(1+d) \\ &\quad + (1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}, \\ (uv)u^{i-2} &\equiv (1+a)^{i-1}(1+c) + \mu(1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}, \\ (vuv)u^{i-3} &\equiv (1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}. \end{aligned}$$

Case 2. If  $i \equiv 1 \pmod{4}$ , then we have

$$\begin{aligned} u^i &\equiv (1+a)^i + \mu(1+a)^{i-2}(1+d) \pmod{I_K^{i+1}(G)}, \\ vu^{i-1} &\equiv (1+a)^{i-1}(1+c) + (1+a)^{i-2}(1+d) \pmod{I_K^{i+1}(G)}, \\ (uv)u^{i-2} &\equiv (1+a)^{i-1}(1+c) \pmod{I_K^{i+1}(G)}, \\ (vuv)u^{i-3} &\equiv (1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}. \end{aligned}$$

Case 3. If  $i \equiv 2 \pmod{4}$ , then we have

$$\begin{aligned} u^i &\equiv (1+a)^i \pmod{I_K^{i+1}(G)}, \\ vu^{i-1} &\equiv (1+a)^{i-1}(1+c) + (1+a)^{i-2}(1+d) \\ &\quad + \mu(1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}, \\ (uv)u^{i-2} &\equiv (1+a)^{i-1}(1+c) + \mu(1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}, \\ (vuv)u^{i-3} &\equiv (1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}. \end{aligned}$$

Case 4. If  $i \equiv 3 \pmod{4}$ , then we have

$$\begin{aligned} u^i &\equiv (1+a)^i + \mu(1+a)^{i-2}(1+d) \pmod{I_K^{i+1}(G)}, \\ vu^{i-1} &\equiv (1+a)^{i-1}(1+c) + (1+a)^{i-2}(1+d) \pmod{I_K^{i+1}(G)}, \\ (uv)u^{i-2} &\equiv (1+a)^{i-1}(1+c) \pmod{I_K^{i+1}(G)}, \\ (vuv)u^{i-3} &\equiv (1+a)^{i-3}(1+c)(1+d) \pmod{I_K^{i+1}(G)}. \end{aligned}$$

It follows that the elements  $\{u^i, vu^{i-1}, uvu^{i-2}, vuvu^{i-3}\}$  are linearly independent modulo  $I_K^{i+1}(G)$ . Therefore, the matrix of decomposition is regular and  $\{1, u^i, vu^j, vuvu^k, uvu^l \mid 0 \leq i, j, k, l\}$  form a filtered multiplicative  $K$ -basis of  $KG$ .

Now let  $G$  be one of the groups  $G_{13}$ ,  $G_{14}$ ,  $G_{18}$ ,  $G_{23}$ ,  $G_{24}$  or  $G_{25}$  with  $m > 5$ . Clearly,  $G$  is two-generated and we can assume that  $u = b_1$ ,  $v = b_2$ , where  $b_1, b_2$  can be written as in (4). Then by (1) we get  $1+d$  is central modulo  $I_K^3(G)$  and

$$(1+c)(1+a) \equiv (1+a)(1+c) + (1+a)^2 + (1+d) \pmod{I_K^3(G)}. \quad (5)$$

It follows that

$$\begin{cases} uv \equiv (\alpha_1 + \alpha_2)\beta_1(1+a)^2 + \Delta(1+a)(1+c) + \alpha_2\beta_1(1+d) & (\text{mod } I_K^3(G)); \\ vu \equiv \alpha_1(\beta_1 + \beta_2)(1+a)^2 + \Delta(1+a)(1+c) + \alpha_1\beta_2(1+d) & (\text{mod } I_K^3(G)); \\ u^2 \equiv \alpha_1(\alpha_1 + \alpha_2)(1+a)^2 + \alpha_1\alpha_2(1+d) & (\text{mod } I_K^3(G)); \\ v^2 \equiv \beta_1(\beta_1 + \beta_2)(1+a)^2 + \beta_1\beta_2(1+d) & (\text{mod } I_K^3(G)). \end{cases}$$

Since the  $K$ -dimension of  $I_K^2(G)/I_K^3(G)$  equals 3 and  $\Delta \neq 0$ , we have that  $uv \not\equiv vu \pmod{I_K^3(G)}$  and  $u^2 \not\equiv v^2 \pmod{I_K^3(G)}$ . Thus either  $v^2 \equiv 0 \pmod{I_K^3(G)}$  and  $\beta_1 = 0$  or  $u^2 \equiv 0 \pmod{I_K^3(G)}$  and  $\alpha_1 = 0$ . It is easy to see that the second case is symmetric to the first one, so we consider only the first one. Therefore,  $u = (1+a) + \mu(1+c)$  and  $v = 1+c$ , where  $\mu = \frac{\alpha_2}{\alpha_1}$ .

We will prove that the elements  $\{(uv)^i u, u^2 v(uv)^{i-1}, (vu)^i v, u^3 (vu)^{i-1}\}$  form a basis modulo  $I_K^{2i+2}(G)$  and the elements  $\{(uv)^i, u^2 (vu)^{i-1}, (vu)^i, u^2 (uv)^{i-1}\}$  form a basis modulo  $I_K^{2i+1}(G)$ , where  $i > 1$ .

First, it is easy to see by induction that

$$\begin{aligned} (uv)^{2i+1} &\equiv (1+a)^{4i+1}(1+c) \pmod{I_K^{4i+3}(G)}, \\ (uv)^{2i+2} &\equiv (1+a)^{4i+3}(1+c) + (1+a)^{4i+1}(1+c)(1+d) \pmod{I_K^{4i+5}(G)}, \\ (vu)^{2i+1} &\equiv (1+a)^{4i+2} + (1+a)^{4i+1}(1+c) \\ &\quad + (1+a)^{4i}(1+d) \pmod{I_K^{4i+3}(G)}, \\ (vu)^{2i+2} &\equiv (1+a)^{4i+4} + (1+a)^{4i+3}(1+c) \\ &\quad + (1+a)^{4i+1}(1+c)(1+d) \pmod{I_K^{4i+5}(G)}. \end{aligned}$$

By (1) and (5) we also have modulo  $I_K^4(G)$  that

$$\begin{aligned} (1+c)(1+a)^2 &\equiv (1+a)^2(1+c), \\ u^2 &\equiv (1+\mu)(1+a)^2 + \mu(1+d), \\ u^2 v &\equiv (1+\mu)(1+a)^2(1+c) + \mu(1+c)(1+d), \\ u^3 &\equiv (1+\mu)(1+a)^3 + \mu(1+\mu)(1+a)^2(1+c) \\ &\quad + \mu(1+a)(1+d) + \mu^2(1+c)(1+d). \end{aligned}$$

Now we consider the following two cases:

Case 1. Let  $k = 2j + 1$  be odd. Then

$$\begin{aligned} (uv)^k u &\equiv (uv)^{2j+1} u \equiv (1+a)^{4j+3} + (1+a)^{4j+2}(1+c) \\ &\quad + (1+a)^{4j+1}(1+d) \pmod{I_K^{4j+4}(G)}, \\ u^2 v (uv)^{k-1} &\equiv u^2 v (uv)^{2(j-1)+2} \equiv (1+\mu)(1+a)^{4j+2}(1+c) \\ &\quad + (1+a)^{4j}(1+c)(1+d) \pmod{I_K^{4j+4}(G)}, \end{aligned}$$

$$\begin{aligned}
(vu)^k v &\equiv (vu)^{2j+1} v \equiv (1+a)^{4j+2}(1+c) \\
&\quad + (1+a)^{4j}(1+c)(1+d) \pmod{I_K^{4j+4}(G)}, \\
u^3(vu)^{k-1} &\equiv u^3(vu)^{2(j-1)+2} \equiv (1+\mu)(1+a)^{4j+3} + (1+\mu)(1+a)^{4j+3}(1+c) \\
&\quad + (1+\mu)(1+a)^{4j}(1+c)(1+d) \\
&\quad + \mu(1+a)^{4j+1}(1+d) \pmod{I_K^{4j+4}(G)}, \\
u^2(vu)^{k-1} &\equiv u^2(vu)^{2(j-1)+2} \equiv (1+\mu)(1+a)^{4j+2} \\
&\quad + (1+\mu)(1+a)^{4j+1}(1+c) + \mu(1+a)^{4j}(1+d) \\
&\quad + (1+a)^{4j-1}(1+c)(1+d) \pmod{I_K^{4j+3}(G)}, \\
(vu)^k &\equiv (vu)^{2j+1} \equiv (1+a)^{4j+2} + (1+a)^{4j+1}(1+c) \\
&\quad + (1+a)^{4j}(1+d) \pmod{I_K^{4j+3}(G)}, \\
u^2(uv)^{k-1} &\equiv u^2(uv)^{2(j-1)+2} \equiv (1+\mu)(1+a)^{4j+1}(1+c) \\
&\quad + (1+a)^{4j-1}(1+c)(1+d) \pmod{I_K^{4j+3}(G)}, \\
(uv)^k &\equiv (uv)^{2j+1} \equiv (1+a)^{4j+1}(1+c) \pmod{I_K^{4j+3}(G)}.
\end{aligned}$$

Case 2. Let  $k = 2j$  be even. Then

$$\begin{aligned}
(uv)^k u &\equiv (uv)^{2(j-1)+2} u \equiv (1+a)^{4j+1} + (1+a)^{4j}(1+c) \\
&\quad + (1+a)^{4j-2}(1+c)(1+d) \pmod{I_K^{4j+2}(G)}, \\
u^2 v(uv)^{k-1} &\equiv u^2 v(uv)^{2(j-1)+1} \equiv (1+\mu)(1+a)^{4j}(1+c) \\
&\quad + (1+a)^{4j-2}(1+c)(1+d) \pmod{I_K^{4j+2}(G)}, \\
u^3(vu)^{k-1} &\equiv u^3(vu)^{2(j-1)+1} \equiv (1+\mu)(1+a)^{4j+1} + (1+\mu)(1+a)^{4j}(1+c) \\
&\quad + \mu(1+\mu)(1+a)^{4j-2}(1+c)(1+d) \\
&\quad + (1+a)^{4j-1}(1+d) \pmod{I_K^{4j+2}(G)}, \\
(vu)^k v &\equiv (vu)^{2(j-1)+2} v \equiv (1+a)^{4j}(1+c) \pmod{I_K^{4j+2}(G)}, \\
(uv)^k &\equiv (uv)^{2(j-1)+2} \equiv (1+a)^{4j-1}(1+c) \\
&\quad + (1+a)^{4j-3}(1+c)(1+d) \pmod{I_K^{4j+1}(G)}, \\
u^2(vu)^{k-1} &\equiv u^2(vu)^{2(j-1)+1} \equiv (1+\mu)(1+a)^{4j} + (1+\mu)(1+a)^{4j-1}(1+c) \\
&\quad + (1+a)^{4j-2}(1+d) \\
&\quad + \mu(1+a)^{4j-3}(1+c)(1+d) \pmod{I_K^{4j+1}(G)},
\end{aligned}$$

$$\begin{aligned}
(vu)^k &\equiv (vu)^{2(j-1)+2} \equiv (1+a)^{4j} + (1+a)^{4j-1}(1+c) \\
&\quad + (1+a)^{4j-3}(1+c)(1+d) \pmod{I_K^{4j+1}(G)}, \\
u^2(uv)^{k-1} &\equiv u^2(uv)^{2(j-1)+1} \equiv (1+\mu)(1+a)^{4j-1}(1+c) \\
&\quad + \mu(1+a)^{4j-3}(1+c)(1+d) \pmod{I_K^{4j+1}(G)}.
\end{aligned}$$

It follows that  $(uv)^k$ ,  $u^2(vu)^{k-1}$ ,  $(vu)^k$ ,  $u^2(uv)^{k-1}$  and also  $(vu)^k u$ ,  $u^2 v(uv)^{k-1}$ ,  $(vu)^k v$ ,  $u^3(vu)^{k-1}$  are linearly independent modulo  $I_K^{2k+1}(G)$  and modulo  $I_K^{2k+1}(G)$ , respectively. Therefore, as before, the matrix of decomposition is regular and

$$\{(uv)^i u, (vu)^i v, (uv)^i, (vu)^i, u^2 v(uv)^j, u^3(vu)^j, u^2(vu)^j, u^2(uv)^j\}$$

form a filtered multiplicative  $K$ -basis of  $KG$ .

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